Unit-V

Sec:29 Local compactness

Definition:

A space X is said to be locally compact at x if there is some compact susbspace C of X that contains a neighbourhood of x. If X is locally compact at each of its point, then X is said to be locally compact.

Theorem: 29.1

Let X be a space. Then X is locally compact Hausdorff space iff there exist a space Y satisfying the following conditions.

(i) X is a subspace of Y

(ii) The set Y-X consist of a single point

(iii) Y is a compact Hausdorff space

If Y and Y' are two space satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Proof:

Step:1

First we prove the uniqueness

Let Y and Y' be two spaces satisfying these conditions.

Define h: $Y \to Y'$ by letting h maps the single point p of Y-X to the point q of Y'-X and letting h equal the identity on X.

We have to show that if U is open in Y then h(Ʊ) is open in Y'.

This implies that h is that h is the homeomorphism.

Case(i)

Suppose p doesn't belongs to Ʊ

Then h(Ʊ) = Ʊ [since h is a identity map]

Since Ʊ is open in Y and it contained in X,Ʊ is open in X

Also since X is open in Y' we have U is open in Y'. Hence h(Ʊ) is open in Y'

Case(ii)

Suppose p ∈ Ʊ

Let C = Y-Ʊ. Then C is closed in Y

Since Y is compact, we have C is a compact subspace of Y.

Since C is contained in X, It is a compact subspace of X. Also since X is a subspace of Y', the space c is also a compact subspace of Y'

∴ C is closed in Y'

∴Y' –C is open in Y'

∴h(Ʊ) is open in Y'

Step: 2

Suppose X is locally compact Hausdorff space. Take some object that is not a point of X denote it by the symbol ∞

Let Y = X∪{∞}

Define a collection of open set of Y to consist of type(i) all sets U that are open in X and type(ii) all sets of the form Y-C, where C is a compact subspace of X ---------------- ①

We shall show that the collection ① is a topology

The empty set is the set of type(i) and space Y is the set of type (ii)

Now, Checking that the intersection of two open sets in open involves 3 cases.

Case(i) $U_1 \cap U_2$ is of type (i)

Case (ii) $(Y-C_1) \cap (Y-C_2) = Y-(C_1 \cap C_2)$ is of type (ii)

Case (iii) $U_1 \cap (Y-C_1) = U_1 \cap (X-C_1)$ is of type (i) because of C_1 is closed in X

Now, We check the union of any collection of open sets is open.

(i) ∪Ʊ α = U is open in X and is of type (i)

(ii) ∪(Y-C β) = Y-∩Cβ = Y-C is of type (ii)

(iii) (∪Ʊα) ∪ (∪(Y – Cβ)) = Ʊ∪ (Y -C) =Y-(C-∪) is of type(ii)

Since C-Ʊ is a closed subspace of C, we have C-Ʊ is compact

Hence ① is a topology on Y

Next, we have to show that X is a subspace of Y , we show its intersection with X is open in X.

If U is of type(i), then Ʊ∩X =Ʊ

If Y-C is of type (ii), then (Y-C) ∩X = X-C

In both cases the sets are open inX.

Conversely,

Any set open in X is a set of type(i) and therefore open in Y

∴ X is a subspace of Y.

Now, we show that Y is compact

Let **A** be an open covering of Y

The collection **A** must contain an open set of type(ii) say Y-C, Since none of the open sets of type (i) contain the point ∞

Take all the members of **A** different from Y-C and intersects them with X, they form a collection of open sets of X covering C.

Since C is compact, finitely many of them cover C

The corresponding finite collection of elements of **A** along with the elements Y-C, cover all of Y

Hence Y is compact

Next, we show that Y is Hausdorff

Let x,y be two points of Y

If both of them lie in x, there are disjoint sets U and V open in X containing x and y respectively. Since X is Hausdroff

On the otherhand, If x∈X, y=∞

We can choose a compact set C is X containing a neighbourhood U of X.

Then U and Y-C are disjoint neighbourhood of x and ∞

Hence Y is Hausdroff.

Step:3

Now, we prove the converse

suppose a space Y satisfying the condition (i),(ii),(iii) exists

Since Y is Hausdroff, X is Hausdroff

Let x ∈X

We have show that X is locally compact at x

Choose disjoint open sets U and V of Y containing x and the single point of Y-X respectively.

Then the set C = Y-V is closed in Y, so it a compact subspace of Y.

Since C lies in X , it is also a compact subspace of X.

Also C contains the neighbourhood U of x.

Hence X is locally compact

∴ X is locally compact Hausdroff.

Definition:

If Y is compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be compactification of X. If Y – X equals a single point, then Y is called the one point compactification of X.

Theorem 29.2

Let X be a Hausdorff space. Then X is locally compact iff given x in X and given a neighbourhood U of x, there is a neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V}$CU

Proof:

Assume that given xϵX and given neighbourhood U of x there is a neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V}$CU.

Since xϵVC$\overline{V}$, the set C=$\overline{V}$ is the required compact subspace of X containing a neighbourhood V of x.

Hence X is locally compact.

Conversely,

Assume that X is locally compact.

Let xϵX.

Let U be a neighbourhood of x.

Take the one point compactification Y of X.

Let C= Y – U .

Then C is closed in Y.

Hence C is compact subspaces of Y (Since Y is compact Hausdroff space)

Choose disjoint open sets V and W containing x and c respectively. Then the closure $\overline{V}$ of V in Y is compact.

Also, $\overline{V}$ is disjoint from C.

We have $\overline{V}$C C^c=U

Thus, $\overline{V}$CU

Corollary 29.3

Let X be locally compact Hausdorff and let A be a subspace of X. A is closed or open in X. Then A is locally compact.

Proof:

Suppose A is closed in X.

To prove A is locally compact.

Let xϵA.

Let C be a compact subspace of X containing the neighbourhood U of x in X. (Since X is locally compact)

Then C∩A contains the neighbourhood U∩A of x in A.

Hence A is locally compact.

Section 28 Limit Point Compactness

Definition:

A space X is said to be limit point compact if every infinite subset of X has a limit point.

Theorem 28.1 Compactness implies limit point compactness, but not conversely.

Proof:

Let X be a compact space. Let A be a subset of X. We have to prove that if A is infinite, then A has a limit point. We prove the this by contra positive method ie) If A has no limit point, then A must be finite.

Suppose A has no limit point. Then A contain all its limit points, so that A is closed. For each a∈A, we can choose a neighbourhood U_a of a such that U_a intersects A in the point a alone. The space X is coveted by the open set X – A and the open sets U_a. Since X is compact, it can be covered by finitely many of this sets. Since X – A does not intersect A and each set U_a contains only one point of A.

Hence the set A must be finite.

Definition:

Let X be a topological space. If (x_n) is a sequence of points of x and if $n_1 < n_2 < \ldots\ldots <$ $n_i < \ldots\ldots$ is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{ni}$ is called a subsequence of the sequence (x_n). The space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence.

Theorem 28.2

Let X be a metrizable space. Then the following conditions are equivalent (i) X is compact (ii) X is limit point compact (iii) X is sequentially compact

Proof:

(i) → (ii)

Let X be a compact space. Let A be a subset of X. We have to prove that if A is infinite, then A has a limit point. We prove the this by contra positive method ie) If A has no limit point, then A must be finite.

Suppose A has no limit point. Then A contain all its limit points, so that A is closed. For each a∈A, we can choose a neighbourhood U_a of a such that U_a intersects A in the point a alone. The space X is coveted by the open set X – A and the open sets U_a. Since X is compact, it can be covered by finitely many of this sets. Since X – A does not intersect A and each set U_a contains only one point of A.

Hence the set A must be finite.

(ii) → (iii)

Assume X is limit point compact. To prove X is sequentially compact. Let sequence (x_n) be a point of X. Consider the set A= $\{x_n \mid n\epsilon Z_+\}$

Case (i): Suppose A is finite

Then there is a point x such that $x=x_n$ for infinitely many values of n. In this case, the sequence (x_n) has a subsequence that is constant and converges obviously.

Case (ii): Suppose A is infinite

Then A has an limit point of x. We define a subsequence of sequence (x_n) coverging to x as follows:

First choose n_1, so that $x_{n_1}\epsilon$ B(x,1). Then choose n_2, so that $x_{n_2}\epsilon$ B(x,$\frac{1}{2}$), and so on.

Then choose n_i, so that $x_{n_i}\epsilon$ B(x, $\frac{1}{i}$), and so on.

Then the subsequence $(x_{n_1}, x_{n_2},\ldots\ldots, x_{n_i},\ldots..) \rightarrow$ x

Thus X is sequentially compact.

(iii)→(i)

Assume that X is sequentially compact.

To prove X is compact.

If X is sequentially compact, then the lebesgue number lemma holds for X. Let A be an open covering of X. We assume that there is no δ>0 such that each set of diameter less than δ has an element of A containing it, and derive a contradiction. (Since, Let A be an open covering of the metric space (X,d). If X is compact, then there is a δ>0 such that for each subset of X having diameter less than δ, there exists a element of A containing it. The number δ is called a lebesgue number for the covering A)

Our assumption implies inparticular that for each positive integer n, there is a set of diameter less than $\frac{1}{n}$ that is not contained in any element of A.

Let C_n be such a set. Choose a point $x_n\epsilon$ C_n, for each n. By hypothesis, some subsequence (x_{n_i}) of the sequence (x_n) converges to the point 'a' (say).

Now, 'a' belongs to some element A of the collection A, because A is open, we may choose an ε>0 such that B(a,ε)contained in A. If i is large enough that $\frac{1}{n_i}<\frac{\varepsilon}{2}$, then the set C_{n_i} lies in the $\frac{\varepsilon}{2}$ neighbourhood of x_{n_i} . If i is also choose an large enough that d(x_{n_i} ,a)< $\frac{\varepsilon}{2}$, then $\frac{\varepsilon}{2}$ lies in the ε neighbourhood of 'a'.

But this means that C_{n_i} contained in A, contrary to hypothesis.

We show that if X is sequentially compact, then given ε>0 there is a finite covering of X by open ε balls.

Once again we proceed by contradiction. Assume that there is an ε>0 such that X cannot be covered by finitely many ε balls. Construct a sequence of points x_n of X as follows; First choose x_1 to . be any point of X. Nothing that the ball B(x_1,ε) is not all of X. (Otherwise X would be covered by a single ε balls); Choose x_2 to be a point of X not in B(x_1,ε); and so on.

In general, given x_1,x_2,……,x_n, choose x_{n+1} to be a point not in the union B(x_1,ε)UB(x_2,ε)U…..UB(x_n,ε), using the fact that these balls donot cover X. Note that by construction d(x_{n+1},x_i)>ε for i=1 to n. Thus the sequence (x_n) have no convergent subsequence, infact, any ball of radius $\frac{\varepsilon}{2}$ that contain x_n atmost one value of n.

Finally, we show that if X is sequentially compact, then X is compact.

Let A be an open covering of X. Since X is sequentially compact, the open covering A has a lebesgue number δ. Let ε=$\frac{\delta}{3}$. Since X is sequentially compact, to find a finite covering of X by open ε balls. Each of these balls has diameter atmost $\frac{2\delta}{3}$, so it lies in an element of A. Choosing one such element of A for each of these ε balls, we obtain a finite subcollection of A that covers X. Hence X is compact.